HAMILTONIAN DYNAMICS OF A CHARGED FLUID, INCLUDING ELECTRO- AND MAGNETOHYDRODYNAMICS

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Nondissipative hydrodynamic equations for a charged, polarized and magnetized nonrelativistic fluid moving in a self-consistent electromagnetic field are presented and shown to possess a hamiltonian structure, associated to the dual of a certain Lie algebra of semidirect-product type. Ideal magnetohydrodynamics and electrohydrodynamics both emerge in hamiltonian form as *regular limits* (i.e., special cases) of the hamiltonian structure for the more general theory.

Dynamics of a charged fluid. The nondissipative, nonrelativistic motion of a charged, polarized and magnetized fluid in the presence of a self-consistent electromagnetic field is described by the following equations in the laboratory galilean frame,

$$\partial_t M_i = -\partial_k \pi_i^k, \qquad i, k = 1, 2, 3, \tag{1a}$$

$$\partial_t \rho = -\operatorname{div}(\rho \upsilon), \qquad \partial_t S = -\operatorname{div}(S \upsilon), \tag{1b, c}$$

$$\partial_t B = -\operatorname{curl} E, \qquad \partial_t D = \operatorname{curl} H - J, \tag{1d, e}$$

$$\operatorname{div} D = q, \qquad \operatorname{div} B = 0, \tag{1f, g}$$

with

$$M_i = \rho \upsilon_i + g_i, \qquad J = q\upsilon, \tag{1h, i}$$

$$\pi_i^k = P\delta_i^k + M_i \upsilon^k - E_i^* D^k - H_i^* B^k. \tag{1j}$$

In (1a–1j) ρ is the mass density of the fluid; υ is the fluid velocity; $\partial_t = \partial/\partial t$ is the partial time derivative; $\partial_k = \partial/\partial x^k$ the partial space derivative; $M = \rho\upsilon + g$ is the total momentum density, including the electromagnetic momentum density, g; π_i^k is the momentum flux tensor, containing P the total pressure, including thermodynamic, electromagnetic, and, possibly, striction contributions; S is the total entropy density; E the electric field intensity; B the magnetic intensity; D the electric displacement vector; H the magnetic induction; q the electric charge density defined

in (1f); and $J = q\upsilon$ the convective current density given in (1i). The dynamical equations (1a–1e) preserve the static Maxwell equations (1f, 1g), provided the latter are assumed to be satisfied initially and the total charge is conserved (i.e., when $\partial_t q + \operatorname{div} J = 0$). Generalization of the system (1) to the case of many fluid species is straightforward, but not discussed here.

In the notation of Pauli [1], p. 101, the vectors in (1j)

$$E^* = E + \upsilon \times B, \qquad H^* = H - \upsilon \times D, \tag{2a, b}$$

measure the forces on a unit electric charge (2a) or magnetic pole (2b) moving with the fluid. That is E^* and H^* are, respectively, the electric field intensity and magnetic induction as measured in the fluid rest frame. For later use in the constitutive relations we also define fields D^* and B^* by (denoting c as the speed of light)

$$D^* = D + \upsilon \times H/c^2, \qquad B^* = B - \upsilon \times E/c^2, \tag{2c, d}$$

which are, respectively, the electric displacement vector and magnetic field intensity as measured in the rest frame of the fluid.

The system of equations (1a–1g) with definitions (1h–1j) and (2a, 2b) is closed by specifying constitutive relations for the quantities P, E^*, H^*, g in (1h–1j), the temperature T, and the chemical potential μ via the thermodynamic identity for the total internal energy density ϵ, in the laboratory galilean frame. This

thermodynamic first law, to first order in v/c, is

$$d\epsilon = T\,dS + \mu\,d\rho + E^* \cdot dD + H^* \cdot dB - g \cdot dv. \qquad (3)$$

The total pressure is defined by

$$P = -\epsilon + \rho\mu + TS + E^* \cdot D + H^* \cdot B, \qquad (4a)$$

which is the thermodynamic potential in the intensive variables,

$$dP = S\,dT + \rho\,d\mu + D \cdot dE^* + B \cdot dH^* + g \cdot dv. \qquad (4b)$$

The total energy density in the laboratory frame is equal to

$$h = \tfrac{1}{2}\rho v^2 + g \cdot v + \epsilon \qquad (5a)$$

$$= -\tfrac{1}{2}\rho v^2 + M \cdot v + \epsilon \qquad \text{[by (1h)]}. \qquad (5b)$$

As an example, we may take the total energy density in (5) to be

$$h' = \tfrac{1}{2}\rho v^2 + \bar{\epsilon}(\rho, S) + \tfrac{1}{2}E \cdot D + \tfrac{1}{2}\dot{H} \cdot B, \qquad (6)$$

and choose the constitutive relations to be linear and isotropic in the rest frame of the fluid, namely

$$D^* = \alpha E^*, \qquad B^* = \beta H^*, \qquad (7)$$

with constants α and β. In this case, one finds (ref. [2], p. 34)

$$E^* = \alpha^{-1}D + (c^{-2}/\alpha\beta)\,v \times B, \qquad (8a)$$

$$H^* = \beta^{-1}B - (c^{-2}/\alpha\beta)\,v \times D. \qquad (8b)$$

Consequently, h' in (6) may be rewritten using (2), (7), and (8) as

$$h' = \tfrac{1}{2}\rho v^2 + \bar{\epsilon}(\rho, S) + \frac{D^2}{2\alpha} + \frac{B^2}{2\beta} + \left(1 - \frac{c^{-2}}{\alpha\beta}\right)v \cdot g$$

$$= \tfrac{1}{2}\rho v^2 + v \cdot g + \bar{\epsilon}(\rho, S) + \frac{D^2}{2\alpha} + \frac{B^2}{2\beta} - \frac{c^{-2}}{\alpha\beta}v \cdot g, \quad (9a)$$

where

$$g = D \times B \qquad (9b)$$

is the Minkowski form [1–3] of the electromagnetic momentum density. The total pressure in this example by (4a), (5), (8), and (9) is found to separate additively as

$$P = P_{\text{MAT}} + \tfrac{1}{2}E^* \cdot D + \tfrac{1}{2}H^* \cdot B, \qquad (10a)$$

where

$$P_{\text{MAT}} = -\bar{\epsilon}(\rho, S) + \rho\mu + TS = \rho\partial(\bar{\epsilon}/\rho)/\partial\rho|_S. \qquad (10b)$$

We now state the main result of this work: *The electromagnetic fluid equations* (1a–1j) *comprise a hamiltonian system*. This system can be written in the form

$$\partial_t F = \{H, F\}, \qquad F \in \{M_i, \rho, S, B, D\}, \qquad (11)$$

with hamiltonian, H, given by

$$H = \int d^3x\, h = \int d^3x\, [-\tfrac{1}{2}\rho v^2 + M \cdot v + \epsilon], \qquad (12)$$

where ϵ satisfies the first law (3). The variational derivatives of H in (12) are determined from

$$\delta H = \int d^3x\, [(\mu - \tfrac{1}{2}v^2)\,\delta\rho + T\,\delta S + v \cdot \delta M$$

$$+ (M - \rho v - g) \cdot \delta v + E^* \cdot \delta D + H^* \cdot \delta B], \qquad (13)$$

so that, e.g., $\delta H/\delta M = v$ and the coefficient of δv vanishes by (1h). The Poisson bracket $\{H, F\}$ in (11) is given by (summing on repeated indices)

$$\{H, F\} = -\int d^3x \left[\frac{\delta F}{\delta\rho}\partial_j\rho\frac{\delta H}{\delta M_j} + \frac{\delta F}{\delta S}\partial_j S\frac{\delta H}{\delta M_j} \right.$$

$$+ \frac{\delta F}{\delta M_i}\left(\rho\partial_i\frac{\delta H}{\delta\rho} + S\partial_i\frac{\delta H}{\delta S} + (\partial_j M_i + M_j\partial_i)\frac{\delta H}{\delta M_j}\right)$$

$$+ (D_j\partial_i - \partial_k D^k\delta_{ij})\frac{\delta H}{\delta D_j} + (B_j\partial_i - \partial_k B^k\delta_{ij})\frac{\delta H}{\delta B_j}\Bigg)$$

$$+ \frac{\delta F}{\delta D_i}(\partial_j D_i - D^k\partial_k\delta_{ij})\frac{\delta H}{\delta M_j}$$

$$+ \frac{\delta F}{\delta B_i}(\partial_j B_i - B^k\partial_k\delta_{ij})\frac{\delta H}{\delta M_j}\Bigg]$$

$$+ \int d^3x \left(\frac{\delta F}{\delta B_i}\epsilon_{ijk}\partial_k\frac{\delta H}{\delta D_j} - \frac{\delta F}{\delta D_i}\epsilon_{ijk}\partial_k\frac{\delta H}{\delta B_j}\right), \quad (14b)$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the totally antisymmetric tensor density in three dimensions.

Substitution of the variational derivatives of H obtained from (13) into the Poisson bracket (14a, b) readily yields the dynamical equations (1a–1e) with π_i^k given by (1j) and P given by (4a).

In particular, we verify (1b–1e) in hamiltonian form directly from (13) and (14) as follows,

$$\partial_t\rho = \{H, \rho\} = -\partial_j(\rho v^j), \qquad (15)$$

$$\partial_t S = \{H, S\} = -\partial_j (Sv^j), \tag{16}$$

$$\partial_t D_i = \{H, D_i\} = -\partial_j (D_i v^j) + D^k \partial_k v_i + \epsilon_{ijk} \partial_j H^{*k} \tag{17a}$$

$$= [\mathrm{curl}(\mathbf{v} \times \mathbf{D})]_i - v_i (\mathrm{div}\, \mathbf{D}) + (\mathrm{curl}\, \mathbf{H})_i$$

$$- [\mathrm{curl}(\mathbf{v} \times \mathbf{D})]_i \quad [\text{by (2b)}] \tag{17b}$$

$$= (\mathrm{curl}\, \mathbf{H})_i - J_i \quad [\text{by (1f) and (1i)}], \tag{17c}$$

$$\partial_t B_i = \{H, B_i\} = -\partial_j (B_i v^j) + B^k \partial_k v_i - \epsilon_{ijk} \partial_j E^{*k} \tag{18a}$$

$$= [\mathrm{curl}(\mathbf{v} \times \mathbf{B})]_i - v_i (\mathrm{div}\, \mathbf{B}) - (\mathrm{curl}\, \mathbf{E})_i$$

$$- [\mathrm{curl}(\mathbf{v} \times \mathbf{B})]_i \quad [\text{by (2a)}] \tag{18b}$$

$$= -(\mathrm{curl}\, \mathbf{E})_i \quad [\text{by (1g)}]. \tag{18c}$$

Finally, we verify (1a) in hamiltonian form.

$$\partial_t M_i = \{H, M_i\} = -\partial_k \left[\left(\rho \frac{\delta H}{\delta \rho} + S \frac{\delta H}{\delta S} + M_j \frac{\delta H}{\delta M_j} + D_j \frac{\delta H}{\delta D_j} \right. \right.$$

$$\left. \left. + B_j \frac{\delta H}{\delta B_j} - h \right) \delta_i^k + M_i \frac{\delta H}{\delta M_k} - D^k \frac{\delta H}{\delta D^i} - B^k \frac{\delta H}{\delta B^i} \right] \tag{19a}$$

$$= -\partial_k [(\rho(-\tfrac{1}{2}v^2 + \mu) + TS + \mathbf{M} \cdot \mathbf{v} + \mathbf{E}^* \cdot \mathbf{D}$$

$$+ \mathbf{H}^* \cdot \mathbf{B} - h) \delta_i^k + M_i v^k - E_i^* D^k - H_i^* B^k]$$

$$[\text{by (13)}] \tag{19b}$$

$$= -\partial_k (P \delta_i^k + M_i v^k - E_i^* D^k - H_i^* B^k)$$

$$[\text{by (4) and (5)}]. \tag{19c}$$

To interpret the momentum equation (19c) we specialize to the example of linear isotropic media with $\mathbf{g} = \mathbf{D} \times \mathbf{B}$ as in (9b). For this case the pressure separates additively as in (10). Using the relation (1h) in the form $\mathbf{M} = \rho \mathbf{v} + \mathbf{D} \times \mathbf{B}$ then decomposes the momentum equation (19c) into an equation for the fluid velocity,

$$\rho \, \mathrm{d}v_i/\mathrm{d}t = -\partial_i P_{\mathrm{MAT}} + q(\mathbf{E} + \mathbf{v} \times \mathbf{B})_i$$

$$+ \tfrac{1}{2} (D^k E_{k,i} - E^k D_{k,i} + B^k H_{k,i} - H^k B_{k,i})$$

$$+ \partial_j [-\mathbf{v} \cdot \mathbf{D} \times \mathbf{B} \, \delta_i^j + v^j (\mathbf{D} \times \mathbf{B})_i + D^j (\mathbf{B} \times \mathbf{v})_i$$

$$+ B^j (\mathbf{v} \times \mathbf{D})_i], \tag{20}$$

with $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{v} \cdot \mathbf{V}$ the material derivative. Eq. (20) contains various terms describing moving-material effects. In particular, the last four terms in the square bracket describe such effects. Although eq. (20) is not simple to interpret, it will be useful later when the equations of magnetohydrodynamics and electrohydrodynamics are derived as special cases of the inclusive theory presented here.

Lie-algebraic structure of the Poisson bracket. The Poisson bracket (14) is bilinear, skew-adjoint, and satisfies the Jacobi identity. (The first two properties are obvious.) We verify the Jacobi identity by associating this Poisson bracket to the dual of the following Lie algebra of semidirect product type. The Poisson bracket (14) is the sum of two parts: a semidirect product part and a two-cocycle. The first part (14a) represents the natural Poisson bracket [4] on the dual to the Lie algebra

$$\mathbf{L}_1 = \mathbf{V} \times [\Lambda^0 \oplus \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^1]. \tag{21}$$

The symbol $\times$ denotes the semidirect product with respect to the natural action of the Lie algebra of vector fields $\mathbf{V}$ on $\mathbf{R}^3$ acting on differential k-forms Λ^k, $k = 0, 1$, and $\oplus$ denotes direct sum. The corresponding commutator for the Lie algebra $\mathbf{L}_1$ is given by [4]

$$[(X; f; g; \theta; \phi), (\bar{X}; \bar{f}; \bar{g}; \bar{\theta}; \bar{\phi})]$$

$$= ([X, \bar{X}] ; X(\bar{f}) - \bar{X}(f); X(\bar{g}) - \bar{X}(g);$$

$$X(\bar{\theta}) - \bar{X}(\theta); X(\bar{\phi}) - \bar{X}(\phi)). \tag{22}$$

Dual coordinates on the Lie algebra $\mathbf{L}_1$ are: $\mathbf{M}$ dual to $X \in \mathbf{V}$; ρ to $f \in \Lambda^0$; S to $g \in \Lambda^0$; $\mathbf{D}$ to $\theta \in \Lambda^1$; and $\mathbf{B}$ to $\phi \in \Lambda^1$). The second part of the bracket (9b) represents the following generalized two-cocycle [5] on $\mathbf{L}_1$:

$$\omega_1((X;f;g;\theta;\phi),\,(\overline{X};\overline{f};\overline{g};\overline{\theta};\overline{\phi}))$$

$$=-\theta^i\epsilon_{ijk}\partial_k\phi^j+\phi^i\epsilon_{ijk}\partial_k\theta^j \tag{23a}$$

$$=-\nu(\theta,\phi)+\nu(\phi,\theta). \tag{23b}$$

Here, ω_1 is skew-symmetric and satsifies

$$\omega_1([(X;\ ),(\overline{X};\ )],(X';\ ))+\text{c.p.}\sim 0, \tag{24}$$

where $(X;\ )$ denotes the full set of quantities in (22), c.p. denotes cyclic permutation of $\{X,\overline{X},X'\}$, and $a\sim b$ for two quantities denotes that the difference $(a-b)$ is the divergence of a vector [5]. The two-co-cycle (23) is a generalized symplectic two-cocycle [6], since with respect to the bilinear form ν in (23) on $\Lambda^1\otimes\Lambda^1$, the action of V on Λ^1 is ν-self-adjoint (see ref. [6], eq. (3.1)).

Having been associated to the dual of Lie algebra L_1 with two-cocycle ω_1, the Poisson bracket in (14) satisfies the Jacobi identity.

The Poisson bracket (14) can be obtained from another Poisson bracket, expressed in terms of ρ, S, M, D, and the vector potential A, via the mapping $B =$ curl A. Namely,

$$\{H,F\}=-\int d^3x\left[\frac{\delta F}{\delta\rho}\partial_j\rho\frac{\delta H}{\delta M_j}+\frac{\delta F}{\delta S}\partial_j S\frac{\delta H}{\delta M_j}\right.$$

$$+\frac{\delta F}{\delta M_i}\left(\rho\partial_i\frac{\delta H}{\delta\rho}+S\partial_i\frac{\delta H}{\delta S}+(\partial_j M_i+M_j\partial_i)\frac{\delta H}{\delta M_j}\right.$$

$$\left.+(D_j\partial_i-\partial_k D^k\partial_{ij})\frac{\delta H}{\delta D_j}+(\partial_j A_i+A_{j,i})\frac{\delta H}{\delta A_j}\right) \tag{25a}$$

$$\left.+\frac{\delta F}{\delta D_i}(\partial_j D_i-D^k\partial_k\delta_{ij})\frac{\delta H}{\delta M_j}+\frac{\delta F}{\delta A_i}(A_j\partial_i+A_{i,j})\frac{\delta H}{\delta M_j}\right]$$

$$+\int d^3x\left(\frac{\delta F}{\delta D_i}\delta_{ij}\frac{\delta H}{\delta A_j}-\frac{\delta F}{\delta A_i}\delta_{ij}\frac{\delta H}{\delta D_j}\right). \tag{25b}$$

The first part (25a) of this Poisson bracket is naturally associated to the dual of the Lie algebra [4]

$$L_2=V\times[\Lambda^0\oplus\Lambda^0\oplus\Lambda^1\oplus\Lambda^2], \tag{26}$$

with commutator and dual coordinates as before, except that for L_2 the vector potential A is dual to $\phi\in\Lambda^2$. (In n dimensions A would be dual to Λ^{n-1} and B to Λ^{n-2}. However, Λ^{n-2} and Λ^{n-1} happen to have the same number of components for $n=3$, so

the same notation, ϕ, can be used for the dual coordinate of A or B in the commutator expression (22) for L_1 or L_2.) The second part (25b) of the Poisson bracket represents the symplectic two-cocycle on L_2:

$$\omega_2((X;f;g;\theta;\phi),\,(\overline{X};\overline{f};\overline{g};\overline{\theta};\overline{\phi}))$$

$$=\theta^i\delta_{ij}\phi^j-\phi^i\delta_{ij}\overline{\theta}^j=\mathbf{\theta}\cdot\overline{\mathbf{\phi}}-\mathbf{\phi}\cdot\overline{\mathbf{\theta}}. \tag{27}$$

(Formula (27) is a symplectic two-cocycle on L_2 since in $\mathbb{R}^3$ $(\Lambda^1)\approx(\Lambda^2)^*$, by proposition 3.5 in ref. [6].) The two-cocyle in (27) maps to the two-cocyle in (23) via $\mathbf{\phi}\to$ curl $\mathbf{\phi}$ and $\overline{\mathbf{\phi}}\to$ curl $\overline{\mathbf{\phi}}$.

The dynamical equation for A found by using the Poisson bracket in (25) is

$$\partial_t A=\{H,A\}=-\nabla(A\cdot\mathbf{v})-E, \tag{28}$$

which yields the magnetic induction equation (1d) upon taking the curl of both sides. The other equations (1a–c, e) remain the same as before when using the Poisson bracket (25).

Magnetohydrodynamics and electrohydrodynamics. Two approximations that can be made directly in the hamiltonian structure (12)–(14) lead to useful simplified models in hamiltonian form. First, if D is absent (so that $H^*=H$, $M=\rho\mathbf{v}$, and g and q are absent) and provided $B=\beta H$ and div $B=0$, we recover the equations of compressible ideal magnetohydrodynamics in hamiltonian form [4,7]. Using the variational derivatives in (13), and either Poisson bracket (14) or (25) with D absent gives

$$\partial_t\rho=\{H,\rho\}=-\text{div}(\rho\mathbf{v}), \tag{29}$$

$$\partial_t S=\{H,S\}=-\text{div}(S\mathbf{v}), \tag{30}$$

$$\partial_t M_i=\{H,M_i\}=-\partial_k(P\delta_i^k+M_iv^k-H_iB^k). \tag{31a}$$

Hence, for the pressure decomposition in (10) we have

$$\rho\,d\mathbf{v}/dt=-\nabla P_{\text{MAT}}-B\times\text{curl}\,H-\tfrac{1}{2}H^2\nabla\beta, \tag{31b}$$

which is the motion equation for ideal magnetohydrodynamics. Finally, using Poisson bracket (14) gives the magnetic field equation,

$$\partial_t B=\{H,B\}=\text{curl}(\mathbf{v}\times B), \tag{32}$$

while using Poisson bracket (25) results in the vector potential dynamics,

$$\partial_t A = \{H, A\} = \nabla(\upsilon \cdot A) + \upsilon \times \mathrm{curl}\, A. \qquad (33)$$

These equations have the well-known feature that the lines of B are "frozen" into the fluid motion [8].

The second approximate model is obtained when B is absent (so that $E^* = E, M = \rho\upsilon$, and g is absent). Then if $E = \alpha^{-1}D$ and div $D = q$, we obtain the equations of compressible ideal electrohydrodynamics in hamiltonian form. By using eqs. (13) and (14) we find

$$\partial_t \rho = \{H, \rho\} = - \mathrm{div}(\rho\upsilon), \qquad (34)$$

$$\partial_t S = \{H, S\} = - \mathrm{div}(S\upsilon), \qquad (35)$$

$$\partial_t M_i = \{H, M_i\} = - \partial_k (P\delta_i^k + \rho\upsilon_i\upsilon^k - E_i D^k). \quad (36a)$$

Hence, using (10) for the pressure decomposition yields

$$\rho\, \mathrm{d}\upsilon/\mathrm{d}t = - \nabla P_{\mathrm{MAT}} + qE - D \times \mathrm{curl}\, E - \tfrac{1}{2}E^2 \nabla\alpha, \qquad (36b)$$

which is the motion equation for ideal electrohydrodynamics when curl $E = 0$. Finally, the field dynamics for the electric displacement vector derives from

$$\partial_t D = \{H, D\} = \mathrm{curl}(\upsilon \times D) - q\upsilon. \qquad (37)$$

As a result of (37) we find that both the lines of D and the charge density q are frozen into the motion of the fluid for electrohydrodynamics. This model also describes newtonian self-gravitating fluid motion, upon setting $q = \rho$ and taking $\alpha^{-1} = 4\pi G$ with G being the gravitational constant, so that $D = - \nabla(\Phi/4\pi G)$, Φ the gravitational potential.

Thus, magnetohydrodynamics and electrohydrodynamics both emerge in hamiltonian form as two regular limits (special cases actually), of the hamiltonian structure (12)–(14) forthe more general equations (1), (2) of a charged fluid interacting self-consistently with the electromagnetic field, including the moving-material effects of induced polarization and magnetization.

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